

Power Flow Equation Formulation

Yousu Chen
PNNL

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This document is a description of how to formulate the power flow equation. The current version is for Newton-Raphson AC power flow.

Basic Equations

We start with the basic equations

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{V} \quad (1)$$

$$I_i = \sum_{k=1}^N Y_{ik} V_k \quad (2)$$

$$\begin{aligned} S_i &= V_i I_i^* \\ &= V_i \left(\sum_{k=1}^N Y_{ik} V_k \right)^* \\ &= V_i \sum_{k=1}^N Y_{ik}^* V_k^* \end{aligned} \quad (3)$$

where N is the number of buses in the network, \mathbf{V} is a vector of voltages, \mathbf{I} is a vector of currents and \mathbf{Y} is the Y matrix.

Suppose we define the complex voltages as

$$V_i = |V_i| e^{j\theta_i} \quad (4)$$

and then define the phase angle between bus i and bus k as

$$\theta_{ik} = \theta_i - \theta_k \quad (5)$$

Further, we can decompose the elements of the Y matrix into real and imaginary parts using

$$Y_{ik} = G_{ik} + jB_{ik} \quad (6)$$

The G_{ik} are called conductances and the B_{ik} are called susceptances.

Using these definitions we can write the expressions for the quantity \mathbf{S} as

$$S_i = \sum_{k=1}^N |V_i||V_k| e^{j\theta_{ik}} (G_{ik} - jB_{ik}) \quad (7)$$

We can further decompose the S_i into real and imaginary parts using

$$S_i = P_i + jQ_i \quad (8)$$

This can be rewritten as

$$P_i + jQ_i = \sum_{k=1}^N |V_i||V_k| (\cos \theta_{ik} + j \sin \theta_{ik}) (G_{ik} - jB_{ik}) \quad (9)$$

Resolving equation (??) into real and imaginary parts gives the expressions

$$\begin{cases} P_i = \sum_{k=1}^N |V_i||V_k| (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}) \\ Q_i = \sum_{k=1}^N |V_i||V_k| (G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}) \end{cases} \quad (10)$$

Newton-Raphson AC Power Flow

Now set up power flow equations in the form of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The voltage magnitude and phase angle of the slack/reference bus (θ_1 and $|V_1|$) are known.

Define

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_2 \\ \cdot \\ \cdot \\ \theta_N \end{bmatrix} \quad (11)$$

$$|\mathbf{V}| = \begin{bmatrix} |V_2| \\ \cdot \\ \cdot \\ |V_N| \end{bmatrix} \quad (12)$$

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\theta} \\ |\mathbf{V}| \end{bmatrix} \quad (13)$$

The form of the vector function $\mathbf{f}(\mathbf{x})$ is

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} P_2(\mathbf{x}) - P_2 \\ \cdot \\ \cdot \\ P_N(\mathbf{x}) - P_N \\ \hline Q_2(\mathbf{x}) - Q_2 \\ \cdot \\ \cdot \\ Q_N(\mathbf{x}) - Q_N \end{bmatrix} = - \begin{bmatrix} \Delta \mathbf{P}(\mathbf{x}) \\ \Delta \mathbf{Q}(\mathbf{x}) \end{bmatrix} = \mathbf{0} \quad (14)$$

where

$$\begin{cases} \Delta \mathbf{P} = P_i - P_i(\mathbf{x}) \\ \Delta \mathbf{Q} = Q_i - Q_i(\mathbf{x}) \end{cases} \quad (15)$$

P_i and Q_i can be obtained from the equation (??).

Now consider the Jacobian of \mathbf{f} , \mathbf{J} . The dimension of \mathbf{J} using the expression for the complex voltage (??) is $(2(N-1))(2(N-1))$. The Jacobian

itself has the form

$$\begin{aligned} \mathbf{J} &= \begin{bmatrix} \mathbf{J1} & \mathbf{J2} \\ \mathbf{J3} & \mathbf{J4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial \mathbf{P}}{\partial \theta} & \frac{\partial \mathbf{P}}{\partial |\mathbf{V}|} \\ \frac{\partial \mathbf{Q}}{\partial \theta} & \frac{\partial \mathbf{Q}}{\partial |\mathbf{V}|} \end{bmatrix} \end{aligned} \quad (16)$$

The N-R iteration itself has the form

$$\mathbf{J} \Delta \mathbf{x} = -\mathbf{f}(\mathbf{x}) \quad (17)$$

Combining equations (??), (??) and (??) gives the expression

$$\begin{bmatrix} \mathbf{J1} & \mathbf{J2} \\ \mathbf{J3} & \mathbf{J4} \end{bmatrix} \begin{bmatrix} \Delta \theta \\ \Delta |\mathbf{V}| \end{bmatrix} = \begin{bmatrix} \Delta \mathbf{P}(\mathbf{x}) \\ \Delta \mathbf{Q}(\mathbf{x}) \end{bmatrix} \quad (18)$$

For the n^{th} iteration, we can rearrange equation (??) to get

$$\begin{bmatrix} \theta^{n+1} \\ |\mathbf{V}|^{n+1} \end{bmatrix} = \begin{bmatrix} \theta^n \\ |\mathbf{V}|^n \end{bmatrix} - [\mathbf{J}(x^n)]^{-1} \begin{bmatrix} \Delta P(x^n) \\ \Delta Q(x^n) \end{bmatrix} \quad (19)$$

The expressions for the elements of \mathbf{J} can be found below:

$$J1 = \begin{cases} \frac{\partial P_i}{\partial \theta_i} = -\sum_{k \neq i} |V_i| |V_k| (G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}) \\ \frac{\partial P_i}{\partial \theta_k} = |V_i| |V_k| (G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}) \quad i \neq k \end{cases} \quad (20)$$

$$J2 = \begin{cases} \frac{\partial P_i}{\partial |V_i|} = 2|V_i| G_{ii} + \sum_{k \neq i} |V_k| (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}) \\ \frac{\partial P_i}{\partial |V_k|} = |V_i| (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}) \quad i \neq k \end{cases} \quad (21)$$

$$J3 = \begin{cases} \frac{\partial Q_i}{\partial \theta_i} = \sum_{k \neq i} |V_i| |V_k| (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}) \\ \frac{\partial Q_i}{\partial \theta_k} = -|V_i| |V_k| (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}) \quad i \neq k \end{cases} \quad (22)$$

$$J4 = \begin{cases} \frac{\partial Q_i}{\partial |V_i|} = -2|V_i| B_{ii} + \sum_{k \neq i} |V_k| (G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}) \\ \frac{\partial Q_i}{\partial |V_k|} = |V_i| (G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}) \quad i \neq k \end{cases} \quad (23)$$

The diagonal elements of $J1 - J4$ can also be written in the following form:

$$J1_{diag} = \frac{\partial P_i}{\partial \theta_i} = -Q_i - B_{ii} |V_i|^2 \quad (24)$$

$$J2_{diag} = \frac{\partial P_i}{\partial |V_i|} = P_i - G_{ii}|V_i|^2 \quad (25)$$

$$J3_{diag} = \frac{\partial Q_i}{\partial \theta_i} = \frac{P_i}{|V_i|} + G_{ii}|V_i| \quad (26)$$

$$J4_{diag} = \frac{\partial Q_i}{\partial |V_i|} = \frac{Q_i}{|V_i|} - B_{ii}|V_i| \quad (27)$$

where P_i and Q_i are given by equation (??).

The solution strategy is:

1. Assign initial voltage magnitudes and zero phase angles (θ) to all buses. For PQ buses (load buses), the voltage magnitudes are set equal to 1.0; for PV buses (voltage-controlled/generator buses), the voltage magnitudes are specified.
2. Calculate P_i , Q_i using equation (??).
3. Calculate ΔP_i , ΔQ_i using equation (??).
4. Calculate the element of Jacobian \mathbf{J} using equation (??) to (??).
5. Solve the linear equation of (??) to find $\Delta \theta$ and $\Delta |V|$.
6. Update new voltage magnitudes and phase angles using equation (??).
7. Continue until the residuals of ΔP and ΔQ are less than the specified tolerance.