

# Weighted-Least-Square(WLS) State Estimation

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November 7, 2014

This document is a description of how to formulate the weighted-least squares (WLS) state estimation problem. Most of the formulation is based on the book by Abur and Exposito<sup>1</sup>.

Power system state estimation is a central component in power system Energy Management Systems. A state estimator receives field measurement data from remote terminal units through data transmission systems, such as a Supervisory Control and Data Acquisition (SCADA) system. Based on a set of non-linear equations relating the measurements and power system states (i.e. bus voltage, and phase angle), a state estimator fine-tunes power system state variables by minimizing the sum of the residual squares. This is the well-known WLS method.

The mathematical formulation of the WLS state estimation algorithm for an  $n$ -bus power system with  $m$  measurements is given below.

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<sup>1</sup>Ali Abur, Antonio Gomez Exposito, "Power System State Estimation Theory and Implementation", CRC Press

## Basic Equations

The starting equation for the WLS state estimation algorithm is

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ z_m \end{bmatrix} = \begin{bmatrix} h_1(x_1, x_2, \dots, x_n) \\ h_2(x_1, x_2, \dots, x_n) \\ \cdot \\ \cdot \\ h_m(x_1, x_2, \dots, x_n) \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_m \end{bmatrix} = h(x) + e \quad (1)$$

The vector  $z$  of  $m$  measured values is

$$z^T = [z_1 \quad z_2 \quad \dots \quad z_m]$$

The vector  $h$

$$h^T = [h_1(x) \quad h_2(x) \quad \dots \quad h_m(x)]$$

containing the non-linear functions  $h_i(x)$  relates the predicted value of measurement  $i$  to the state vector  $x$  containing  $n$  variables

$$x^T = [x_1 \quad x_2 \quad \dots \quad x_n]$$

and  $e$  is the vector of measurement errors

$$e^T = [e_1 \quad e_2 \quad \dots \quad e_m]$$

The measurement errors  $e_i$  are assumed to satisfy the following statistical properties. First, the errors have zero mean

$$E(e_i) = 0, i = 1, \dots, m \quad (2)$$

Second, the errors are assumed to be independent, ( $E[e_i e_j] = 0$  for  $i \neq j$ ), such that the covariance matrix is diagonal

$$Cov(e) = E(e \cdot e^T) = R = diag\{\sigma_1^2, \sigma_2^2, \dots, \sigma_m^2\} \quad (3)$$

The objective function is then given by the relations

$$J(x) = \sum_{i=1}^m (z_i - h_i(x))^2 / R_{ii} = [z - h(x)]^T R^{-1} [z - h(x)] \quad (4)$$

The minimization condition is

$$g(x) = \frac{\partial J(x)}{\partial x} = -H^T(x)R^{-1}[z - h(x)] = 0 \quad (5)$$

where  $H(x) = \partial h(x)/\partial x$ . Expanding  $g(x)$  into its Taylor series leads to the expression

$$g(x) = g(x^k) + G(x^k)(x - x^k) + \dots = 0 \quad (6)$$

where the  $k + 1$  iterate is related to the  $k^{\text{th}}$  iterate via

$$x^{k+1} = x^k - G(x^k)^{-1}g(x^k)$$

and  $G(x^k)$  is the gain matrix

$$G(x^k) = \frac{\partial g(x^k)}{\partial x} = H^T(x^k)R^{-1}H(x^k)$$

Note the each iterate  $g(x^k)$  still satisfies

$$g(x^k) = -H^T(x^k)R^{-1}(z - h(x^k))$$

## The Normal Equation and solution procedure:

The normal equation for the state estimation calculation follows from equation (??) and is given by the expression

$$G(x^k)\Delta x^{k+1} = H^T(x^k)R^{-1}(z - h(x^k)) \quad (7)$$

where  $\Delta x^{k+1} = x^{k+1} - x^k$ . The WLS SE algorithm is based on this equation and consists of the following steps

1. Set  $k = 0$
2. Initialize the state vector  $x^k$ , typical a flat start
3. Calculate the measurement function  $h(x^k)$
4. Build the measurement Jacobian  $H(x^k)$
5. Calculate the gain matrix of  $G(x^k) = H^T(x^k)R^{-1}H(x^k)$
6. Calculate the RHS of the normal equation  $H^T(x^k)R^{-1}(z - h(x^k))$
7. Solve the normal equation ?? for  $\Delta x^k$
8. Check for convergence using  $\max |\Delta x^k| \leq \epsilon$
9. If not converged, update  $x^{k+1} = x^k + \Delta x^k$  and go to 3. Otherwise stop

## The measurement function $h(x^k)$

The measured quantities and their relation to the state variables are listed below

1. Real and reactive power injection at bus  $i$

$$P_i = V_i \sum_{i \neq j} V_j (G_{ij} \cos \theta_{ij} + B_{ij} \sin \theta_{ij}) \quad (8)$$

$$Q_i = V_i \sum_{i \neq j} V_j (G_{ij} \sin \theta_{ij} + B_{ij} \cos \theta_{ij}) \quad (9)$$

2. Real and reactive power flow from bus  $i$  to bus  $j$ :

$$P_{ij} = V_i^2 (g_{si} + g_{ij}) - V_i V_j (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij}) \quad (10)$$

$$Q_{ij} = -V_i^2 (b_{si} + b_{ij}) - V_i V_j (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij}) \quad (11)$$

3. Line current flow magnitude from bus  $i$  to bus  $j$ :

$$I_{ij} = \sqrt{P_{ij}^2 + Q_{ij}^2} / V_i \quad (12)$$

These functions represent the set of functions  $h_i(x)$  that relate the state variables  $V_i$  and  $\theta_i$  to the measurements. The variables in the above expressions are defined as

- $V_i$  is the voltage magnitude at bus  $i$
- $\theta_i$  is the phase angle at bus  $i$
- $\theta_{ij} = \theta_i - \theta_j$
- $G_{ij} + jB_{ij}$  is the  $ij$ th element of the Y-bus matrix
- $g_{ij} + jb_{ij}$  is the admittance of the series branch between bus  $i$  and bus  $j$
- $g_{si} + jb_{sj}$  is the admittance of the shunt branch at bus  $i$

## The measurement Jacobian $H$ :

The Jacobian matrix  $H$  can be written as

$$H = \begin{bmatrix} \frac{\partial P_{inj}}{\partial \theta} & \frac{\partial P_{inj}}{\partial V} \\ \frac{\partial P_{flow}}{\partial \theta} & \frac{\partial P_{flow}}{\partial V} \\ \frac{\partial Q_{inj}}{\partial \theta} & \frac{\partial Q_{inj}}{\partial V} \\ \frac{\partial Q_{flow}}{\partial \theta} & \frac{\partial Q_{flow}}{\partial V} \\ \frac{\partial I_{mag}}{\partial \theta} & \frac{\partial I_{mag}}{\partial V} \\ 0 & \frac{\partial V_{mag}}{\partial V} \end{bmatrix} \quad (13)$$

where the expressions for each block are

$$\frac{\partial P_i}{\partial \theta_i} = \sum_{j=1}^N V_i V_j (-G_{ij} \sin \theta_{ij} + B_{ij} \cos \theta_{ij}) - V_i^2 B_{ii} \quad (14)$$

$$\frac{\partial P_i}{\partial \theta_j} = V_i V_j (G_{ij} \sin \theta_{ij} - B_{ij} \cos \theta_{ij}) \quad (15)$$

$$\frac{\partial P_i}{\partial V_i} = \sum_{j=1}^N V_j (G_{ij} \cos \theta_{ij} + B_{ij} \sin \theta_{ij}) + V_i G_{ii} \quad (16)$$

$$\frac{\partial P_i}{\partial V_j} = V_i (G_{ij} \cos \theta_{ij} + B_{ij} \sin \theta_{ij}) \quad (17)$$

$$\frac{\partial Q_i}{\partial \theta_i} = \sum_{j=1}^N V_i V_j (G_{ij} \cos \theta_{ij} + B_{ij} \sin \theta_{ij}) - V_i^2 G_{ii} \quad (18)$$

$$\frac{\partial Q_i}{\partial \theta_j} = V_i V_j (-G_{ij} \cos \theta_{ij} - B_{ij} \sin \theta_{ij}) \quad (19)$$

$$\frac{\partial Q_i}{\partial V_i} = \sum_{j=1}^N V_j (G_{ij} \sin \theta_{ij} - B_{ij} \cos \theta_{ij}) - V_i B_{ii} \quad (20)$$

$$\frac{\partial Q_i}{\partial V_j} = V_i (G_{ij} \sin \theta_{ij} - B_{ij} \cos \theta_{ij}) \quad (21)$$

$$\frac{\partial P_{ij}}{\partial \theta_i} = V_i V_j (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij}) \quad (22)$$

$$\frac{\partial P_{ij}}{\partial \theta_j} = -V_i V_j (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij}) \quad (23)$$

$$\frac{\partial P_{ij}}{\partial V_i} = -V_j (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij}) + 2(g_{ij} + g_{si}) V_i \quad (24)$$

$$\frac{\partial P_{ij}}{\partial V_j} = -V_i (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij}) \quad (25)$$

$$\frac{\partial Q_{ij}}{\partial \theta_i} = -V_i V_j (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij}) \quad (26)$$

$$\frac{\partial Q_{ij}}{\partial \theta_j} = V_i V_j (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij}) \quad (27)$$

$$\frac{\partial Q_{ij}}{\partial V_i} = -V_j (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij}) - 2(b_{ij} + b_{si}) V_i \quad (28)$$

$$\frac{\partial Q_{ij}}{\partial V_j} = -V_i (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij}) \quad (29)$$

$$\frac{\partial V_i}{\partial V_i} = 1 \quad (30)$$

$$\frac{\partial V_i}{\partial V_j} = 0 \quad (31)$$

$$\frac{\partial V_i}{\partial \theta_i} = 0 \quad (32)$$

$$\frac{\partial V_i}{\partial \theta_j} = 0 \quad (33)$$

$$\frac{\partial I_{ij}}{\partial \theta_i} = \frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} V_i V_j \sin \theta_{ij} \quad (34)$$

$$\frac{\partial I_{ij}}{\partial \theta_j} = -\frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} V_i V_j \sin \theta_{ij} \quad (35)$$

$$\frac{\partial I_{ij}}{\partial V_i} = \frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} (V_i - V_j \cos \theta_{ij}) \quad (36)$$

$$\frac{\partial I_{ij}}{\partial V_j} = \frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} (V_j - V_i \cos \theta_{ij}) \quad (37)$$