Weighted-Least-Square(WLS) State Estimation

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This document is a description of how to formulate the weighted-least squares (WLS) state estimation problem. Most of the formulation is based on the book by Abur and Exposito[1].

Power system state estimation is a central component in power system Energy Management Systems. A state estimator receives field measurement data from remote terminal units through data transmission systems, such as a Supervisory Control and Data Acquisition (SCADA) system. Based on a set of non-linear equations relating the measurements and power system states (i.e. bus voltage, and phase angle), a state estimator fine-tunes power system state variables by minimizing the sum of the residual squares. This is the well-known WLS method.

The mathematical formulation of the WLS state estimation algorithm for an n-bus power system with m measurements is given below.

[1]Ali Abur, Antonio Gomez Exposito, "Power System State Estimation Theroy and Implementation", CRC Press

Basic Equations

The starting equation for the WLS state estimation algorithm is

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_m \end{bmatrix} = \begin{bmatrix} h_1(x_1, x_2, ..., x_n) \\ h_2(x_1, x_2, ..., x_n) \\ \cdot \\ \cdot \\ \cdot \\ h_m(x_1, x_2, ..., x_n) \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_m \end{bmatrix} = h(x) + e \tag{1}$$

The vector z of m measured values is

$$z^T = \begin{bmatrix} z_1 & z_2 & & ... & z_m \end{bmatrix}$$

The vector h

$$h^T = \begin{bmatrix} h_1(x) & h_2(x) & & ... & h_m(x) \end{bmatrix}$$

containing the non-linear functions h_i(x) relates the predicted value of measurement i to the state vector x containing n variables

$$x^T = \begin{bmatrix} x_1 & x_2 & ... & x_n \end{bmatrix}$$

and e is the vector of measurement errors

$$e^T = \begin{bmatrix} e_1 & e_2 & ... & e_m \end{bmatrix}$$

The measurement errors e_i are assumed to satisfy the following statistical properties. First, the errors have zero mean

$$E(e_i) = 0, i = 1, ..., m \tag{2}$$

Second, the errors are assumed to be independent, ($E[e_i e_j] = 0$ for $i \neq j$), such that the covariance matrix is diagonal

$$Cov(e) = E(e \cdot e^T) = R = diag\{\sigma_1^2, \sigma_2^2, ..., \sigma_m^2\} \tag{3}$$

The objective function is then given by the relations

$$J(x) = \sum_{i=1}^{m} (z_i - h_i(x))^2 / R_{ii} = [z - h(x)]^T R^{-1} [z - h(x)] \tag{4}$$

The minimization condition is

$$g(x) = \frac{\partial J(x)}{\partial x} = -H^T(x)R^{-1}[z - h(x)] = 0 \tag{5}$$

where $H(x) = \partial h(x)/\partial x$. Expanding $g(x)$ into its Taylor series leads to the expression

$$g(x) = g(x^k) + G(x^k)(x - x^k) + ... = 0 \tag{6}$$

where the $k+1$ iterate is related to the k^{th} iterate via

$$x^{k+1} = x^k - G(x^k)^{-1} g(x^k)$$

and $G(x^k)$ is the gain matrix

$$G(x^k) = \frac{\partial g(x^k)}{\partial x} = H^T(x^k)R^{-1}H(x^k)$$

Note the each iterate $g(x^k)$ still satisfies

$$g(x^k) = -H^T(x^k)R^{-1}(z - h(x^k))$$

The Normal Equation and solution procedure:

The normal equation for the state estimation calculation follows from equation (6) and is given by the expression

$$G(x^k)\Delta x^{k+1} = H^T(x^k)R^{-1}(z - h(x^k)) \tag{7}$$

where $\Delta x^{k+1} = x^{k+1} - x^k$. The WLS SE algorithm is based on this equation and consists of the following steps

1. Set $k = 0$
2. Initialize the state vector x^k, typical a flat start
3. Calculate the measurement function $h(x^k)$
4. Build the measurement Jacobian $H(x^k)$
5. Calculate the gain matrix of $G(x^k) = H^T(x^k)R^{-1}H(x^k)$
6. Calculate the RHS of the normal equation $H^T(x^k)R^{-1}(z - h(x^k))$
7. Solve the normal equation 7 for Δx^k
8. Check for convergence using $\max \left|\Delta x^k\right| \leq \epsilon$
9. If not converged, update $x^{k+1} = x^k + \Delta x^k$ and go to 3. Otherwise stop

The measurement function $h(x^k)$

The measured quantities and their relation to the state variables are listed below

1. Real and reactive power injection at bus i

$$P_i = V_i \sum_{i \neq j} V_j (G_{ij} \cos\theta_{ij} + B_{ij} \sin\theta_{ij}) \tag{8}$$

$$Q_i = V_i \sum_{i \neq j} V_j (G_{ij} \sin\theta_{ij} + B_{ij} \cos\theta_{ij}) \tag{9}$$

2. Real and reactive power flow from bus i to bus j:

$$P_{ij} = V_i^2 (g_{si} + g_{ij}) - V_i V_j (g_{ij} \cos\theta_{ij} + b_{ij} \sin\theta_{ij}) \tag{10}$$

$$Q_{ij} = -V_i^2 (b_{si} + b_{ij}) - V_i V_j (g_{ij} \sin\theta_{ij} - b_{ij} \cos\theta_{ij}) \tag{11}$$

3. Line current flow magnitude from bus i to bus j:

$$I_{ij} = \sqrt{P_{ij}^2 + Q_{ij}^2} / V_i \tag{12}$$

These functions represent the set of functions $h_i(x)$ that relate the state variables V_i and θ_i to the measurements. The variables in the above expressions are defined as

- V_i is the voltage magnitude at bus i
- θ_i is the phase angle at bus i
- $\theta_{ij} = \theta_i - \theta_j$
- $G_{ij} + jB_{ij}$ is the ijth element of the Y-bus matrix
- $g_{ij} + jb_{ij}$ is the admittance of the series branch between bus i and bus j
- $g_{si} + jb_{sj}$ is the admittance of the shunt branch at bus i

The measurement Jacobian H:

The Jacobian matrix H can be written as

$$H = \begin{bmatrix} \frac{\partial P_{inj}}{\partial \theta} & \frac{\partial P_{inj}}{\partial V} \\ \frac{\partial P_{flow}}{\partial \theta} & \frac{\partial P_{flow}}{\partial V} \\ \frac{\partial Q_{inj}}{\partial \theta} & \frac{\partial Q_{inj}}{\partial V} \\ \frac{\partial Q_{flow}}{\partial \theta} & \frac{\partial Q_{flow}}{\partial V} \\ \frac{\partial I_{mag}}{\partial \theta} & \frac{\partial I_{mag}}{\partial V} \\ 0 & \frac{\partial V_{mag}}{\partial V} \end{bmatrix} \tag{13}$$

where the expressions for each block are

$$\frac{\partial P_i}{\partial \theta_i} = \sum_{j=1}^{N} V_i V_j (-G_{ij} \sin \theta_{ij} + B_{ij} \cos \theta_{ij}) - V_i^2 B_{ii} \tag{14}$$

$$\frac{\partial P_i}{\partial \theta_j} = V_i V_j (G_{ij} \sin \theta_{ij} - B_{ij} \cos \theta_{ij}) \tag{15}$$

$$\frac{\partial P_i}{\partial V_i} = \sum_{j=1}^{N} V_j (G_{ij} \cos \theta_{ij} + B_{ij} \sin \theta_{ij}) + V_i G_{ii} \tag{16}$$

$$\frac{\partial P_i}{\partial V_j} = V_i (G_{ij} \cos \theta_{ij} + B_{ij} \sin \theta_{ij}) \tag{17}$$

$$\frac{\partial Q_i}{\partial \theta_i} = \sum_{j=1}^{N} V_i V_j (G_{ij} \cos \theta_{ij} + B_{ij} \sin \theta_{ij}) - V_i^2 G_{ii} \tag{18}$$

$$\frac{\partial Q_i}{\partial \theta_j} = V_i V_j (-G_{ij} \cos \theta_{ij} - B_{ij} \sin \theta_{ij}) \tag{19}$$

$$\frac{\partial Q_i}{\partial V_i} = \sum_{j=1}^{N} V_j (G_{ij} \sin \theta_{ij} - B_{ij} \cos \theta_{ij}) - V_i B_{ii} \tag{20}$$

$$\frac{\partial Q_i}{\partial V_j} = V_i (G_{ij} \sin \theta_{ij} - B_{ij} \cos \theta_{ij}) \tag{21}$$

$$\frac{\partial P_{ij}}{\partial \theta_i} = V_i V_j (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij}) \tag{22}$$

$$\frac{\partial P_{ij}}{\partial \theta_j} = -V_i V_j (g_{ij} \sin\theta_{ij} - b_{ij} \cos\theta_{ij}) \tag{23}$$

$$\frac{\partial P_{ij}}{\partial V_i} = -V_j (g_{ij} \cos\theta_{ij} + b_{ij} \sin\theta_{ij}) + 2(g_{ij} + g_{si}) V_i \tag{24}$$

$$\frac{\partial P_{ij}}{\partial V_j} = -V_i (g_{ij} \cos\theta_{ij} + b_{ij} \sin\theta_{ij}) \tag{25}$$

$$\frac{\partial Q_{ij}}{\partial \theta_i} = -V_i V_j (g_{ij} \cos\theta_{ij} + b_{ij} \sin\theta_{ij}) \tag{26}$$

$$\frac{\partial Q_{ij}}{\partial \theta_j} = V_i V_j (g_{ij} \cos\theta_{ij} + b_{ij} \sin\theta_{ij}) \tag{27}$$

$$\frac{\partial Q_{ij}}{\partial V_i} = -V_j (g_{ij} \sin\theta_{ij} - b_{ij} \cos\theta_{ij}) - 2(b_{ij} + b_{si}) V_i \tag{28}$$

$$\frac{\partial Q_{ij}}{\partial V_j} = -V_i (g_{ij} \sin\theta_{ij} - b_{ij} \cos\theta_{ij}) \tag{29}$$

$$\frac{\partial V_i}{\partial V_i} = 1 \tag{30}$$

$$\frac{\partial V_i}{\partial V_j} = 0 \tag{31}$$

$$\frac{\partial V_i}{\partial \theta_i} = 0 \tag{32}$$

$$\frac{\partial V_i}{\partial \theta_j} = 0 \tag{33}$$

$$\frac{\partial I_{ij}}{\partial \theta_i} = \frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} V_i V_j \sin\theta_{ij} \tag{34}$$

$$\frac{\partial I_{ij}}{\partial \theta_j} = -\frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} V_i V_j \sin\theta_{ij} \tag{35}$$

$$\frac{\partial I_{ij}}{\partial V_i} = \frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} (V_i - V_j \cos\theta_{ij}) \tag{36}$$

$$\frac{\partial I_{ij}}{\partial V_i} = \frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} (V_j - V_i \cos\theta_{ij}) \tag{37}$$